A locally compact quantum group of triangular matrices.

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Dedicated to Professor M.L. Gorbachuk on the occasion of his 70-th anniversary.

Abstract

We construct a one parameter deformation of the group of 2×2 upper triangular matrices with determinant 1 using the twisting construction. An interesting feature of this new example of a locally compact quantum group is that the Haar measure is deformed in a non-trivial way. Also, we give a complete description of the dual C^*-algebra and the dual comultiplication.

1 Introduction

In [3, 14], M. Enock and the second author proposed a systematic approach to the construction of non-trivial Kac algebras by twisting. To illustrate it, consider a cocommutative Kac algebra structure on the group von Neumann algebra $M = \mathcal{L}(G)$ of a non commutative locally compact (l.c.) group G with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ (here λ_g is the left translation by $g \in G$). Let us define on M another, "twisted", comultiplication $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^*$, where Ω is a unitary from $M \otimes M$ verifying certain 2-cocycle condition, and construct in this way new, non cocommutative, Kac algebra structure on M. In order to find such an Ω, let us, following to M. Rieffel [10] and M. Landstad [8], take an inclusion $\alpha : L^\infty(\hat{K}) \to M$, where $\hat{K}$ is the dual to some abelian subgroup K of G such that $\delta|_K = 1$, where $\delta(\cdot)$ is the module of G. Then, one lifts a usual 2-cocycle Ψ of $\hat{K}$: $\Omega = (\alpha \otimes \alpha)\Psi$. The main result of [3], [14] is that the integral by the Haar measure of G gives also the Haar measure of the deformed object. Recently P. Kasprzak studied the deformation of l.c. groups by twisting in [5], and also in this case the Haar measure was not deformed.

In [4], the authors extended the twisting construction in order to cover the case of non-trivial deformation of the Haar measure. The aim of the present paper is to illustrate this construction on a concrete example and to compute

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explicitly all the ingredients of the twisted quantum group including the dual C^*-algebra and the dual comultiplication. We twist the group von Neumann algebra $\mathcal{L}(G)$ of the group G of 2×2 upper triangular matrices with determinant 1 using the abelian subgroup $K = \mathbb{C}^*$ of diagonal matrices of G and a one parameter family of bicharacters on K. In this case, the subgroup K is not included in the kernel of the modular function of G, this is why the Haar measure is deformed. We compute the new Haar measure and show that the dual C^*-algebra is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ such that

$$\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \quad \hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha},$$

where $q > 0$. Moreover, the comultiplication $\hat{\Delta}$ is given by

$$\hat{\Delta}_t(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha},\ \hat{\Delta}_t(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} \dot{+} \hat{\beta} \otimes \hat{\alpha}^{-1},$$

where $\dot{+}$ means the closure of the sum of two operators.

This paper in organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we present in detail our example computing all the ingredients associated. This example is inspired by [5], but an important difference is that in the present example the Haar measure is deformed in a non trivial way. Finally, we collect some useful results in the Appendix.

2 Preliminaries

2.1 Notations

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H, $\otimes$ the tensor product of Hilbert spaces, von Neumann algebras or minimal tensor product of C^*-algebras, and Σ (resp., σ) the flip map on it. If H, K and L are Hilbert spaces and $X \in B(H\otimes L)$ (resp., $X \in B(H\otimes K), X \in B(K\otimes L)$), we denote by X_{13} (resp., $X_{12},\ X_{23}$) the operator $(1\otimes\Sigma^*)(X\otimes 1)(1\otimes\Sigma)$ (resp., $X\otimes 1,\ 1\otimes X$) defined on $H\otimes K\otimes L$. For any subset X of a Banach space E, we denote by $\langle X\rangle$ the vector space generated by X and $[X]$ the closed vector space generated by X. All l.c. groups considered in this paper are supposed to be second countable, all Hilbert spaces are separable and all von Neumann algebras have separable preduals.

Given a *normal semi-finite faithful* (n.s.f.) weight θ on a von Neumann algebra M (see [12]), we denote: $\mathcal{M}_\theta^+ = \{x \in M^+ \mid \theta(x) < +\infty\}$, $\mathcal{N}_\theta = \{x \in M \mid x^*x \in M_\theta^+\}$, and $\mathcal{M}_\theta = \langle\mathcal{M}_\theta^+\rangle$.

When A and B are C^*-algebras, we denote by $\mathrm{M}(A)$ the algebra of the multipliers of A and by $\mathrm{Mor}(A, B)$ the set of the morphisms from A to B.

2.2 G-products and their deformation

For the notions of an action of a l.c. group G on a C^*-algebra A, a C^* dynamical system (A, G, α), a crossed product $G\,{}_\alpha\!\ltimes A$ of A by G see [9]. The crossed product has the following universal property:

For any C^*-covariant representation (π, u, B) of (A, G, α) (here B is a C^*-algebra, $\pi : A \to B$ a morphism, u is a group morphism from G to the unitaries of $M(B)$, continuous for the strict topology), there is a unique morphism $\rho \in \mathrm{Mor}(G\,{}_\alpha \ltimes A, B)$ such that

$$\rho(\lambda_t) = u_t, \quad \rho(\pi_\alpha(x)) = \pi(x) \quad \forall t \in G, x \in A.$$

Definition 1 *Let G be a l.c. abelian group, B a C^*-algebra, λ a morphism from G to the unitary group of $M(B)$, continuous in the strict topology of $M(B)$, and θ a continuous action of $\hat{G}$ on B. The triplet (B, λ, θ) is called a G-product if $\theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g$ for all $\gamma \in \hat{G}$, $g \in G$.*

The unitary representation $\lambda : G \to \mathrm{M}(B)$ generates a morphism :

$$\lambda \in \mathrm{Mor}(C^*(G), B).$$

Identifying $C^*(G)$ with $C_0(\hat{G})$, one gets a morphism $\lambda \in \mathrm{Mor}(C_0(\hat{G}), B)$ which is defined in a unique way by its values on the characters

$$u_g = (\gamma \mapsto \langle \gamma, g \rangle) \in C_b(\hat{G}) : \ \lambda(u_g) = \lambda_g, \quad \text{for all } g \in G.$$

One can check that λ is injective.

The action θ is done by: $\theta_\gamma(\lambda(u_g)) = \theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g = \lambda(u_g(. - \gamma))$. Since the u_g generate $C_b(\hat{G})$, one deduces that:

$$\theta_\gamma(\lambda(f)) = \lambda(f(. - \gamma)), \quad \text{for all } f \in C_b(\hat{G}).$$

The following definition is equivalent to the original definition by Landstad [8] (see [5]):

Definition 2 *Let (B, λ, θ) be a G-poduct and $x \in M(B)$. One says that x verifies the Landstad conditions if*

$$\left\{ \begin{array}{ll} (i) & \theta_\gamma(x) = x, \quad \textit{for any} \quad \gamma \in \hat{G}, \\ (ii) & \textit{the application } g \mapsto \lambda_g x \lambda_g^* \textit{ is continuous}, \\ (iii) & \lambda(f) x \lambda(g) \in B, \quad \textit{for any} \quad f, g \in C_0(\hat{G}). \end{array} \right. \tag{1}$$

The set $A \in \mathrm{M}(B)$ verifying these conditions is a C^*-algebra called *the Landstad algebra* of the G-product (B, λ, θ). Definition 2 implies that if $a \in A$, then $\lambda_g a \lambda_g^* \in A$ and the map $g \mapsto \lambda_g a \lambda_g^*$ is continuous. One gets then an action of G on A.

One can show that the inclusion $A \to \mathrm{M}(B)$ is a morphism of C^*-algebras, so $\mathrm{M}(A)$ can be also included into $\mathrm{M}(B)$. If $x \in \mathrm{M}(B)$, then $x \in \mathrm{M}(A)$ if and only if

$$\left\{ \begin{array}{ll} (i) & \theta_\gamma(x) = x, \ \text{for all } \gamma \in \hat{G}, \\ (ii) & \text{for all } a \in A, \ \text{the application } g \mapsto \lambda_g x \lambda_g^* a \text{ is continuous.} \end{array} \right. \tag{2}$$

Let us note that two first conditions of (1) imply (2).

The notions of G-product and crossed product are closely related. Indeed, if (A, G, α) is a C^*-dynamical system with G abelian, let $B = G\,{}_\alpha\!\ltimes A$ be the crossed product and λ the canonical morphism from G into the unitary group of $\mathrm{M}(B)$, continuous in the strict topology, and $\pi \in \mathrm{Mor}(A, B)$ the canonical morphism of C^*-algebras. For $f \in \mathcal{K}(G, A)$ and $\gamma \in \hat{G}$, one defines $(\theta_\gamma f)(t) = \overline{\langle \gamma, t \rangle} f(t)$. One shows that θ_γ can be extended to the automorphisms of B in such a way that $(B, \hat{G}, \theta)$ would be a C^*-dynamical system. Moreover, (B, λ, θ) is a G-product and the associated Landstad algebra is $\pi(A)$. θ is called *the dual action*. Conversely, if (B, λ, θ) is a G-product, then one shows that there exists a C^*-dynamical system (A, G, α) such that $B = G\,{}_\alpha\!\ltimes A$. It is unique (up to a covariant isomorphism), A is the Landstad algebra of (B, λ, θ) and α is the action of G on A given by $\alpha_t(x) = \lambda_t x \lambda_t^*$.

Lemma 1 *[5] Let (B, λ, θ) be a G-product and $V \subset A$ be a vector subspace of the Landstad algebra such that:*

- $\lambda_g V \lambda_g^* \subset V$*, for any* $g \in G$*,*
- $\lambda(C_0(\hat{G})) V \lambda(C_0(\hat{G}))$ *is dense in* B*.*

Then V is dense in A.

Let (B, λ, θ) be a G-product, A its Landstad algebra, and Ψ a continuous bicharacter on $\hat{G}$. For $\gamma \in \hat{G}$, the function on $\hat{G}$ defined by $\Psi_\gamma(\omega) = \Psi(\omega, \gamma)$ generates a family of unitaries $\lambda(\Psi_\gamma) \in \mathrm{M}(B)$. The bicharacter condition implies:

$$\theta_\gamma(U_{\gamma_2}) = \lambda(\Psi_{\gamma_2}(. - \gamma_1)) = \overline{\Psi(\gamma_1, \gamma_2)} U_{\gamma_2}, \quad \forall \gamma_1, \gamma_2 \in \hat{G}.$$

One gets then a new action θ^Ψ of $\hat{G}$ on B:

$$\theta^\Psi_\gamma(x) = U_\gamma \theta(x) U_\gamma^*.$$

Note that, by commutativity of G, one has:

$$\theta^\Psi_\gamma(\lambda_g) = U_\gamma \theta(\lambda_g) U_\gamma^* = \overline{\langle \gamma, g \rangle} \lambda_g, \quad \forall \gamma \in \hat{G}, g \in G.$$

The triplet $(B, \lambda, \theta^\Psi)$ is then a G-product, called a *deformed* G-product.

2.3 Locally compact quantum groups [6], [7]

A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when

- M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism which is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (i.e., (M, Δ) is a Hopf-von Neumann algebra).
- There exist n.s.f. weights φ and ψ on M such that
  - φ is left invariant in the sense that $\varphi\big((\omega \otimes \mathrm{id})\Delta(x)\big) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}^+_\varphi$ and $\omega \in M^+_*$,

– ψ is right invariant in the sense that $\psi\big((\mathrm{id}\otimes\omega)\Delta(x)\big) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

Left and right invariant weights are unique up to a positive scalar.

Let us represent M on the GNS Hilbert space of φ and define a unitary W on $H \otimes H$ by

$$W^*(\Lambda(a)\otimes\Lambda(b)) = (\Lambda\otimes\Lambda)(\Delta(b)(a\otimes 1)), \quad \text{for all } a, b \in N_\phi .$$

Here, Λ denotes the canonical GNS-map for φ, $\Lambda\otimes\Lambda$ the similar map for $\varphi\otimes\varphi$. One proves that W satisfies the *pentagonal equation*: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that W is a *multiplicative unitary*. The von Neumann algebra M and the comultiplication on it can be given in terms of W respectively as

$$M = \{(\mathrm{id}\otimes\omega)(W) \mid \omega \in B(H)_*\}^{-\sigma-strong*}$$

and $\Delta(x) = W^*(1\otimes x)W$, for all $x \in M$. Next, the l.c. quantum group (M,Δ) has an antipode S, which is the unique σ-strongly* closed linear map from M to M satisfying $(\mathrm{id}\otimes\omega)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)_*$ and $S(\mathrm{id}\otimes\omega)(W) = (\mathrm{id}\otimes\omega)(W^*)$ and such that the elements $(\mathrm{id}\otimes\omega)(W)$ form a σ-strong* core for S. S has a polar decomposition $S = R\tau_{-i/2}$, where R (the unitary antipode) is an anti-automorphism of M and τ_t (the scaling group of (M,Δ)) is a strongly continuous one-parameter group of automorphisms of M. We have $\sigma(R\otimes R)\Delta = \Delta R$, so φR is a right invariant weight on (M,Δ) and we take $\psi := \varphi R$.

Let σ_t be the modular automorphism group of φ. There exist a number $\nu > 0$, called the scaling constant, such that $\psi\,\sigma_t = \nu^{-t}\,\psi$ for all $t \in \mathbb{R}$. Hence (see [13]), there is a unique positive, self-adjoint operator δ_M affiliated to M, such that $\sigma_t(\delta_M) = \nu^t\,\delta_M$ for all $t \in \mathbb{R}$ and $\psi = \varphi_{\delta_M}$. It is called the modular element of (M,Δ). If $\delta_M = 1$ we call (M,Δ) unimodular. The scaling constant can be characterized as well by the relative invariance $\varphi\,\tau_t = \nu^{-t}\,\varphi$.

For the dual l.c. quantum group $(\hat{M},\hat{\Delta})$ we have :

$$\hat{M} = \{(\omega\otimes\mathrm{id})(W) \mid \omega \in B(H)_*\}^{-\sigma-strong*}$$

and $\hat{\Delta}(x) = \Sigma W(x\otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. A left invariant n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ can be constructed explicitly and the associated multiplicative unitary is $\hat{W} = \Sigma W^*\Sigma$.

Since $(\hat{M},\hat{\Delta})$ is again a l.c. quantum group, let us denote its antipode by $\hat{S}$, its unitary antipode by $\hat{R}$ and its scaling group by $\hat{\tau}_t$. Then we can construct the dual of $(\hat{M},\hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$. The bidual l.c. quantum group $(\hat{\hat{M}},\hat{\hat{\Delta}})$ is isomorphic to (M,Δ).

M is commutative if and only if (M,Δ) is generated by a usual l.c. group $G : M = L^\infty(G), (\Delta_G f)(g,h) = f(gh), (S_G f)(g) = f(g^{-1}),\ \varphi_G(f) = \int f(g)\,dg$, where $f \in L^\infty(G),\ g,h \in G$ and we integrate with respect to the left Haar measure dg on G. Then ψ_G is given by $\psi_G(f) = \int f(g^{-1})\,dg$ and δ_M by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

$L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and $(W_G\xi)(g,h) = \xi(g, g^{-1}h)$, for all $\xi \in H\otimes H = L^2(G\times G)$. Then $\hat{M} = \mathcal{L}(G)$ is the group von Neumann

algebra generated by the left translations $(\lambda_g)_{g\in G}$ of G and $\hat{\Delta}_G(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}_G^{op} := \sigma \circ \hat{\Delta}_G = \hat{\Delta}_G$, so $\hat{\Delta}_G$ is cocommutative.

(M, Δ) is a Kac algebra (see [2]) if $\tau_t = \mathrm{id}$, for all t, and δ_M is affiliated with the center of M. In particular, this is the case when $M = L^\infty(G)$ or $M = \mathcal{L}(G)$.

We can also define the C^*-algebra of continuous functions vanishing at infinity on (M, Δ) by

$$A = [(\mathrm{id} \otimes \omega)(W) \mid \omega \in \mathcal{B}(H)_*]$$

and the reduced C^*-algebra (or dual C^*-algebra) of (M, Δ) by

$$\hat{A} = [(\omega \otimes \mathrm{id})(W) \mid \omega \in \mathcal{B}(H)_*] \,.$$

In the group case we have $A = C_0(G)$ and $\hat{A} = C_r(G)$. Moreover, we have $\Delta \in \mathrm{Mor}(A, A \otimes A)$ and $\hat{\Delta} \in \mathrm{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$.

A l.c. quantum group is called compact if $\varphi(1_M) < \infty$ and discrete if its dual is compact.

2.4 Twisting of locally compact quantum groups [4]

Let (M, Δ) be a locally compact quantum group and Ω a unitary in $M \otimes M$. We say that Ω is a 2-cocycle on (M, Δ) if

$$(\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \Delta)(\Omega).$$

As an example we can consider $M = L^\infty(G)$, where G is a l.c. group, with Δ_G as above, and $\Omega = \Psi(\cdot, \cdot) \in L^\infty(G \times G)$ a usual 2-cocycle on G, i.e., a mesurable function with values in the unit circle $\mathbb{T} \subset \mathbb{C}$ verifying

$$\Psi(s_1, s_2)\Psi(s_1 s_2, s_3) = \Psi(s_2, s_3)\Psi(s_1, s_2 s_3), \text{ for almost all } s_1, s_2, s_3 \in G.$$

This is the case for any measurable bicharacter on G.

When Ω is a 2-cocycle on (M, Δ), one can check that $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^*$ is a new coassociative comultiplication on M. If (M, Δ) is discrete and Ω is any 2-cocycle on it, then (M, Δ_Ω) is again a l.c. quantum group (see [1], finite-dimensional case was treated in [14]). In the general case, one can proceed as follows. Let $\alpha : (L^\infty(G), \Delta_G) \to (M, \Delta)$ be an inclusion of Hopf-von Neumann algebras, i.e., a faithful unital normal *-homomorphism such that $(\alpha\otimes\alpha)\circ\Delta_G = \Delta \circ \alpha$. Such an inclusion allows to construct a 2-cocycle of (M, Δ) by lifting a usual 2-cocycle of $G : \Omega = (\alpha \otimes \alpha)\Psi$. It is shown in [3] that if the image of α is included into the centralizer of the left invariant weight φ, then φ is also left invariant for the new comultiplication Δ_Ω.

In particular, let G be a non commutative l.c. group and K a closed abelian subgroup of G. By Theorem 6 of [11], there exists a faithful unital normal *-homomorphism $\hat{\alpha} : \mathcal{L}(K) \to \mathcal{L}(G)$ such that

$$\hat{\alpha}(\lambda_g^K) = \lambda_g, \quad \text{for all } g \in K, \quad \text{and} \quad \hat{\Delta} \circ \hat{\alpha} = (\hat{\alpha} \otimes \hat{\alpha}) \circ \hat{\Delta}_K,$$

where λ^K and λ are the left regular representation of K and G respectively, and $\hat{\Delta}_K$ and $\hat{\Delta}$ are the comultiplications on $\mathcal{L}(K)$ and $\mathcal{L}(G)$ repectively. The composition of $\hat{\alpha}$ with the canonical isomorphism $L^\infty(\hat{K}) \simeq \mathcal{L}(K)$ given by the Fourier tranformation, is a faithful unital normal *-homomorphism $\alpha : L^\infty(\hat{K}) \to \mathcal{L}(G)$ such that $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_{\hat{K}}$, where $\Delta_{\hat{K}}$ is the comultiplication on $L^\infty(\hat{K})$. The left invariant weight on $\mathcal{L}(G)$ is the Plancherel weight for which

$$\sigma_t(x) = \delta_G^{it} x \delta_G^{-it}, \quad \text{for all } x \in \mathcal{L}(G),$$

where δ_G is the modular function of G. Thus, $\sigma_t(\lambda_g) = \delta_G^{it}(g)\lambda_g$ or

$$\sigma_t \circ \alpha(u_g) = \alpha(u_g(\cdot - \gamma_t)),$$

where $u_g(\gamma) = \langle \gamma, g \rangle,\ g \in G, \gamma \in \hat{G},\ \gamma_t$ is the character K defined by $\langle \gamma_t, g \rangle = \delta_G^{-it}(g)$. By linearity and density we obtain:

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(\hat{K}).$$

This is why we do the following assumptions. Let (M, Δ) be a l.c. quantum group, G an abelian l.c. group and $\alpha : (L^\infty(G), \Delta_G) \to (M, \Delta)$ an inclusion of Hopf-von Neumann algebras. Let φ be the left invariant weight, σ_t its modular group, S the antipode, R the unitary antipode, τ_t the scaling group. Let $\psi = \varphi \circ R$ be the right invariant weight and $\sigma_t^{'}$ its modular group. Also we denote by δ the modular element of (M, Δ). Suppose that there exists a continuous group homomorphism $t \mapsto \gamma_t$ from $\mathbb{R}$ to G such that

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(G).$$

Let Ψ be a continuous bicharacter on G. Notice that $(t, s) \mapsto \Psi(\gamma_t, \gamma_s)$ is a continuous bicharacter on $\mathbb{R}$, so there exists $\lambda > 0$ such that $\Psi(\gamma_t, \gamma_s) = \lambda^{ist}$. We define:

$$u_t = \lambda^{i\frac{t^2}{2}} \alpha\left(\Psi(., -\gamma_t)\right) \quad \text{and} \quad v_t = \lambda^{i\frac{t^2}{2}} \alpha\left(\Psi(-\gamma_t, .)\right).$$

The 2-cocycle equation implies that u_t is a σ_t-cocyle and v_t is a $\sigma_t^{'}$-cocycle. The Connes' Theorem gives two n.s.f. weights on M, φ_Ω and ψ_Ω, such that

$$u_t = [D\varphi_\Omega : D\varphi]_t \quad \text{and} \quad v_t = [D\psi_\Omega : D\psi]_t.$$

The main result of [4] is as follows:

Theorem 1 (M, Δ_Ω) *is a l.c. quantum group with left and right invariant weight* φ_Ω *and* ψ_Ω *respectively. Moreover, denoting by a subscript or a superscript* Ω *the objects associated with* (M, Δ_Ω) *one has:*

- $\tau_t^\Omega = \tau_t$,
- $\nu_\Omega = \nu$ *and* $\delta_\Omega = \delta A^{-1} B$,

- $\mathcal{D}(S_\Omega) = \mathcal{D}(S)$ *and, for all* $x \in \mathcal{D}(S)$, $S_\Omega(x) = uS(x)u^*$.

Remark that, because Ψ is a bicharacter on G, $t \mapsto \alpha(\Psi(.,-\gamma_t))$ is a representation of $\mathbb{R}$ in the unitary group of M and there exists a positive self-adjoint operator A affiliated with M such that

$$\alpha(\Psi(.,-\gamma_t)) = A^{it}, \quad \text{for all } t \in \mathbb{R}.$$

We can also define a positive self-adjoint operator B affiliated with M such that

$$\alpha(\Psi(-\gamma_t, .)) = B^{it}.$$

We obtain :

$$u_t = \lambda^{i\frac{t^2}{2}} A^{it}, \quad v_t = \lambda^{i\frac{t^2}{2}} B^{it}.$$

Thus, we have $\varphi_\Omega = \varphi_A$ and $\psi_\Omega = \psi_B$, where φ_A and ψ_B are the weights defined by S. Vaes in [13].

One can also compute the dual C^*-algebra and the dual comultiplication. We put:

$$L_\gamma = \alpha(u_\gamma), \quad R_\gamma = JL_\gamma J, \quad \text{for all } \gamma \in \hat{G}.$$

From the representation $\gamma \mapsto L_\gamma$ we get the unital *-homomorphism $\lambda_L : L^\infty(G) \to M$ and from the representation $\gamma \mapsto R_\gamma$ we get the unital normal *-homomorphism $\lambda_R : L^\infty(G) \to M^{'}$. Let $\hat{A}$ be the reduced C^*-algebra of (M, Δ). We can define an action of $\hat{G}^2$ on $\hat{A}$ by

$$\alpha_{\gamma_1,\gamma_2}(x) = L_{\gamma_1} R_{\gamma_2} x R^*_{\gamma_2} L^*_{\gamma_1}.$$

Let us consider the crossed product C^*-algebra $B = \hat{G}^2 {}_\alpha\ltimes \hat{A}$. We will denote by λ the canonical morphism from $\hat{G}^2$ to the unitary group of $M(B)$ continuous in the strict topology on M(B), $\pi \in$Mor$(\hat{A}, B)$ the canonical morphism and θ the dual action of G^2 on B. Recall that the triplet $(\hat{G}^2, \lambda, \theta)$ is a $\hat{G}^2$-product. Let us denote by $(\hat{G}^2, \lambda, \theta^\Psi)$ the $\hat{G}^2$-product obtained by deformation of the $\hat{G}^2$-product $(\hat{G}^2, \lambda, \theta)$ by the bicharacter $\omega(g,h,s,t) := \overline{\Psi(g,s)}\Psi(h,t)$ on G^2.

The dual deformed action θ^Ψ is done by

$$\theta^\Psi_{(g_1,g_2)}(x) = U_{g_1} V_{g_2} \theta_{(g_1,g_2)}(x) U^*_{g_1} V^*_{g_2}, \quad \text{for any } g_1, g_2 \in G,\ x \in B,$$

where $U_g = \lambda_L(\Psi^*_g)$, $V_g = \lambda_R(\Psi_g)$, $\Psi_g(h) = \Psi(h,g)$.

Considering Ψ_g as an element of $\hat{G}$, we get a morphism from G to $\hat{G}$, also noted Ψ, such that $\Psi(g) = \Psi_g$. With these notations, one has $U_g = u_{(\Psi(-g),0)}$ and $V_g = u_{(0,\Psi(g))}$. Then the action θ^Ψ on $\pi(\hat{A})$ is done by

$$\theta^\Psi_{(g_1,g_2)}(\pi(x)) = \pi(\alpha_{(\Psi(-g_1),\Psi(g_2))}(x)). \tag{3}$$

Let us consider the Landstad algebra A^Ψ associated with this $\hat{G}^2$-product. By definition of α and the universality of the crossed product we get a morphism

$$\rho \in \mathrm{Mor}(B, \mathcal{K}(H)), \quad \rho(\lambda_{\gamma_1,\gamma_2}) = L_{\gamma_1} R_{\gamma_2} \quad \text{et} \quad \rho(\pi(x)) = x. \tag{4}$$

It is shown in [4] that $\rho(A^\Psi) = \hat{A}_\Omega$ and that ρ is injective on A^Ψ. This gives a canonical isomorphism $A^\Psi \simeq \hat{A}_\Omega$. In the sequel we identify A^Ψ with $\hat{A}_\Omega$. The comultiplication can be described in the following way. First, one can show that, using universality of the crossed product, there exists a unique morphism $\Gamma \in \mathrm{Mor}(B, B \otimes B)$ such that:

$$\Gamma \circ \pi = (\pi \otimes \pi) \circ \hat{\Delta} \quad \text{and} \quad \Gamma(\lambda_{\gamma_1,\gamma_2}) = \lambda_{\gamma_1,0} \otimes \lambda_{0,\gamma_2}.$$

Then we introduce the unitary $\Upsilon = (\lambda_R \otimes \lambda_L)(\tilde{\Psi}) \in \mathrm{M}(B \otimes B)$, where $\tilde{\Psi}(g,h) = \Psi(g, gh)$. This allows us to define the *-morphism $\Gamma_\Omega(x) = \Upsilon\Gamma(x)\Upsilon^*$ from B to $\mathrm{M}(B \otimes B)$. One can show that $\Gamma_\Omega \in \mathrm{Mor}(A^\Psi, A^\Psi \otimes A^\Psi)$ is the comultiplication on A^Ψ.

Note that if $M = \mathcal{L}(G)$ and K is an abelian closed subgroup of G, the action α of K^2 on $C_0(G)$ is the left-right action.

3 Twisting of the group of 2×2 upper triangular matrices with determinant 1

Consider the following subgroup of $SL_2(\mathbb{C})$:

$$G := \left\{ \begin{pmatrix} z & \omega \\ 0 & z^{-1} \end{pmatrix} \ , \ z \in \mathbb{C}^* \ , \ \omega \in \mathbb{C} \right\}.$$

Let $K \subset G$ be the subgroup of diagonal matrices in G, i.e. $K = \mathbb{C}^*$. The elements of G will be denoted by (z, ω), $z \in \mathbb{C}$, $\omega \in \mathbb{C}^*$. The modular function of G is

$$\delta_G((z,\omega)) = |z|^{-2}.$$

Thus, the morphism $(t \mapsto \gamma_t)$ from $\mathbb{R}$ to $\widehat{\mathbb{C}^*}$ is given by

$$\langle \gamma_t, z\rangle = |z|^{2it}, \quad \text{for all } z \in \mathbb{C}^*, t \in \mathbb{R}.$$

We can identify $\widehat{\mathbb{C}^*}$ with $\mathbb{Z} \times \mathbb{R}_+^*$ in the following way:

$$\mathbb{Z} \times \mathbb{R}_+^* \to \widehat{\mathbb{C}^*}, \quad (n, \rho) \mapsto \gamma_{n,\rho} = (re^{i\theta} \mapsto e^{i \ln r \ln \rho} e^{in\theta}).$$

Under this identification, γ_t is the element $(0, e^t)$ of $\mathbb{Z} \times \mathbb{R}_+^*$. For all $x \in \mathbb{R}$, we define a bicharacter on $\mathbb{Z} \times \mathbb{R}_+^*$ by

$$\Psi_x((n,\rho),(k,r)) = e^{ix(k \ln \rho - n \ln r)}.$$

We denote by (M_x, Δ_x) the twisted l.c. quantum group. We have:

$$\Psi_x((n,\rho), \gamma_t^{-1}) = e^{ixtn} = u_{e^{ixt}}((n,\rho)).$$

In this way we obtain the operator A_x deforming the Plancherel weight:

$$A_x^{it} = \alpha(u_{e^{ixt}}) = \lambda^G_{(e^{itx},0)}.$$

In the same way we compute the operator B_x deforming the Plancherel weight:

$$B_x^{it} = \lambda^G_{(e^{-ixt},0)} = A_x^{-it}.$$

Thus, we obtain for the modular element :

$$\delta_x^{it} = A_x^{-it} B_x^{it} = \lambda^G_{(e^{-2itx},0)}.$$

The antipode is not deformed. The scaling group is trivial but, if $x \neq 0$, (M_x, Δ_x) is not a Kac algebra because δ_x is not affiliated with the center of M. Let us look if (M_x, Δ_x) can be isomorphic for different values of x. One can remark that, since $\Psi_{-x} = \Psi_x^*$ is antisymmetric and Δ is cocommutative, we have $\Delta_{-x} = \sigma\Delta_x$, where σ is the flip on $\mathcal{L}(G) \otimes \mathcal{L}(G)$. Thus, $(M_{-x}, \Delta_{-x}) \simeq (M_x, \Delta_x)^{\mathrm{op}}$, where "op" means the opposite quantum group. So, it suffices to treat only strictly positive values of x. The twisting deforms only the comultiplication, the weights and the modular element. The simplest invariant distinguishing the (M_x, Δ_x) is then the specter of the modular element. Using the Fourier transformation in the first variable, on has immediately $\mathrm{Sp}(\delta_x) = q_x^{\mathbb{Z}} \cup \{0\}$, where $q_x = e^{-2x}$. Thus, if $x \neq y,\ x > 0, y > 0$, one has $q_x^{\mathbb{Z}} \neq q_y^{\mathbb{Z}}$ and, consequently, (M_x, Δ_x) and (M_y, Δ_y) are non isomorphic.

We compute now the dual C^*-algebra. The action of K^2 on $C_0(G)$ can be lifted to its Lie algebra $\mathbb{C}^2$. The lifting does not change the result of the deformation (see [5], Proposition 3.17) but simplify calculations. The action of $\mathbb{C}^2$ on $C_0(G)$ will be denoted by ρ. One has

$$\rho_{z_1,z_2}(f)(z,\omega) = f(e^{z_2-z_1}z, e^{-(z_1+z_2)}\omega). \tag{5}$$

The group $\mathbb{C}$ is self-dual, the duality is given by

$$(z_1, z_2) \mapsto \exp\left(i\mathrm{Im}(z_1 z_2)\right).$$

The generators u_z, $z \in \mathbb{C}$, of $C_0(\mathbb{C})$ are given by

$$u_z(w) = \exp\left(i\mathrm{Im}(zw)\right), \quad z, w \in \mathbb{C}.$$

Let $x \in \mathbb{R}$. We will consider the following bicharacter on $\mathbb{C}$:

$$\Psi_x(z_1, z_2) = \exp\left(ix\mathrm{Im}(z_1\overline{z}_2)\right).$$

Let B be the crossed product C^*-algebra $\mathbb{C}^2 \ltimes C_0(G)$. We denote by $((z_1, z_2) \mapsto \lambda_{z_1,z_2})$ the canonical group homomorphism from G to the unitary group of $\mathrm{M}(B)$, continuous for the strict topology, and $\pi \in \mathrm{Mor}(C_0(G), B)$ the canonical homomorphism. Also we denote by $\lambda \in \mathrm{Mor}(C_0(G^2), B)$ the morphism given by the representation $((z_1, z_2) \mapsto \lambda_{z_1,z_2})$. Let θ be the dual action of $\mathbb{C}^2$ on B. We have, for all $z, w \in \mathbb{C}$, $\Psi_x(w, z) = u_{x\overline{z}}(w)$. The deformed dual action is given by

$$\theta^{\Psi_x}_{z_1,z_2}(b) = \lambda_{-x\overline{z}_1,x\overline{z}_2}\theta_{z_1,z_2}(b)\lambda^*_{-x\overline{z}_1,x\overline{z}_2}. \tag{6}$$

Recall that

$$\theta^{\Psi_x}_{z_1,z_2}(\lambda(f)) = \theta_{z_1,z_2}(\lambda(f)) = \lambda(f(\cdot - z_1, \cdot - z_2)), \quad \forall f \in C_b(\mathbb{C}^2). \tag{7}$$

Let $\hat{A}_x$ be the associated Landstad algebra. We identify $\hat{A}_x$ with the reduced C^*-algebra of (M_x, Δ_x). We will now construct two normal operators affiliated with $\hat{A}_x$, which generate $\hat{A}_x$. Let a and b be the coordinate functions on G, and $\alpha = \pi(a)$, $\beta = \pi(b)$. Then α and β are normal operators, affiliated with B, and one can see, using (5), that

$$\lambda_{z_1,z_2}\alpha\lambda^*_{z_1,z_2} = e^{z_2-z_1}\alpha, \quad \lambda_{z_1,z_2}\beta\lambda^*_{z_1,z_2} = e^{-(z_1+z_2)}\beta. \tag{8}$$

We can deduce, using (6), that

$$\theta^{\Psi_x}_{z_1,z_2}(\alpha) = e^{x(\overline{z}_1+\overline{z}_2)}\alpha \ , \quad \theta^{\Psi_x}_{z_1,z_2}(\beta) = e^{x(\overline{z}_1-\overline{z}_2)}\beta. \tag{9}$$

Let T_l and T_r be the infinitesimal generators of the left and right shift respectively, i.e. T_l and T_r are normal, affiliated with B, and

$$\lambda_{z_1,z_2} = \exp\left(i\mathrm{Im}(z_1T_l)\right)\exp\left(i\mathrm{Im}(z_2T_r)\right), \quad \text{for all} \quad z_1, z_2 \in \mathbb{C}.$$

Thus, we have:

$$\lambda(f) = f(T_l, T_r), \quad \text{for all } f \in C_b(\mathbb{C}^2).$$

Let $U = \lambda(\Psi_x)$, we define the following normal operators affiliated with B:

$$\hat{\alpha} = U^*\alpha U \ , \quad \hat{\beta} = U\beta U^*.$$

Proposition 1 *The operators $\hat{\alpha}$ and $\hat{\beta}$ are affiliated with $\hat{A}_x$ and generate $\hat{A}_x$.*

Proof. First let us show that $f(\hat{\alpha}), f(\hat{\beta}) \in \mathrm{M}(\hat{A}_t)$, for all $f \in C_0(\mathbb{C})$. One has, using (7):

$$\begin{aligned}
\theta^{\Psi_x}_{z_1,z_2}(U) &= \lambda\left(\Psi_x(.-z_1, .-z_2)\right)\\
&= Ue^{ix\mathrm{Im}(-\overline{z}_2T_l)}e^{ix\mathrm{Im}(\overline{z}_1T_r)}\Psi_x(z_1,z_2)\\
&= U\lambda_{-x\overline{z}_2,x\overline{z}_1}\Psi_x(z_1,z_2).
\end{aligned}$$

Now, using (9) and (8), we obtain:

$$\theta^{\Psi_x}_{z_1,z_2}(\hat{\alpha}) = \hat{\alpha}, \quad \theta^{\Psi_x}_{z_1,z_2}(\hat{\beta}) = \hat{\beta}, \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Thus, for all $f \in C_0(\mathbb{C})$, $f(\hat{\alpha})$ and $f(\hat{\beta})$ are fixed points for the action θ^{Ψ_x}. Let $f \in C_0(\mathbb{C})$. Using (8) we find:

$$\begin{aligned}
\lambda_{z_1,z_2}f(\hat{\alpha})\lambda^*_{z_1,z_2} &= U^*f(e^{z_2-z_1}\alpha)U,\\
\lambda_{z_1,z_2}f(\hat{\beta})\lambda^*_{z_1,z_2} &= U^*f(e^{-(z_1+z_2)}\beta)U.
\end{aligned} \tag{10}$$

Because f is continuous and vanish at infinity, the applications

$$(z_1,z_2) \mapsto \lambda_{z_1,z_2}f(\hat{\alpha})\lambda^*_{z_1,z_2} \quad \text{and} \quad (z_1,z_2) \mapsto \lambda_{z_1,z_2}f(\hat{\beta})\lambda^*_{z_1,z_2}$$

are norm-continuous and $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$.

Taking in mind Proposition 4 (see Appendix), in order to show that $\hat{\alpha}$ is affiliated with $\hat{A}_x$, it suffices to show that the vector space $\mathcal{I}$ generated by $f(\hat{\alpha})a$, with $f \in C_0(\mathbb{C})$ and $a \in \hat{A}_x$, is dense in $\hat{A}_x$. Using (10), we see that $\mathcal{I}$ is globally invariant under the action implemented by λ. Let $g(z) = (1+\overline{z}z)^{-1}$. As $\lambda(C_0(\mathbb{C}^2))U = \lambda(C_0(\mathbb{C}^2))$, we can deduce that the closure of $\lambda(C_0(\mathbb{C}^2))g(\hat{\alpha})\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is equal to

$$\left[\lambda(C_0(\mathbb{C}^2))(1+\alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2))\right].$$

As the set $U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is dense in B and α is affiliated with B, the set $\lambda(C_0(\mathbb{C}^2))(1+\alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is dense in B. Moreover, it is included in $\lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2))$, so $\lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2))$ is dense in B. We conclude, using Lemma 1, that $\mathcal{I}$ is dense in $\hat{A}_x$. One can show in the same way that $\hat{\beta}$ is affiliated with $\hat{A}_x$.

Now, let us show that $\hat{\alpha}$ and $\hat{\beta}$ generate $\hat{A}_x$. By Proposition 5, it suffices to show that

$$\mathcal{V} = \left\langle f(\hat{\alpha})g(\hat{\beta}),\ f, g \in C_0(\mathbb{C})\right\rangle$$

is a dense vector subspace of $\hat{A}_x$. We have shown above that the elements of $\mathcal{V}$ satisfy the two first Landstad's conditions. Let

$$\mathcal{W} = \left[\lambda(C_0(\mathbb{C}^2))\mathcal{V}\lambda(C_0(\mathbb{C}^2))\right].$$

We will show that $\mathcal{W} = B$. This proves that the elements of $\mathcal{V}$ satisfy the third Landstad's condition, and then $\mathcal{V} \subset \hat{A}_x$. Then (10) shows that $\mathcal{V}$ is globally invariant under the action implemented by λ, so $\mathcal{V}$ is dense in $\hat{A}_x$ by Lemma 1. One has:

$$\mathcal{W} = \left[xU^*f(\alpha)U^2g(\beta)U^*y\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right].$$

Because U is unitary, we can substitute x with xU and y with Uy without changing $\mathcal{W}$:

$$\mathcal{W} = \left[xf(\alpha)U^2g(\beta)y\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right].$$

Using, for all $f \in C_0(\mathbb{C})$, the norm-continuity of the application

$$(z_1,z_2) \mapsto \lambda_{z_1,z_2}f(\alpha)\lambda^*_{z_1,z_2} = e^{z_2-z_1}\alpha,$$

one deduces that

$$\begin{aligned}&\left[f(\alpha)x\,,\ f \in C_0(\mathbb{C}),\, x \in \lambda(C_0(\mathbb{C}^2))\right]\\ &= \left[xf(\alpha)\,,\ f \in C_0(\mathbb{C}),\, x \in \lambda(C_0(\mathbb{C}^2))\right].\end{aligned}$$

In particular,

$$\mathcal{W} = \left[f(\alpha)xU^2g(\beta)y\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right].$$

Now we can commute $g(\beta)$ and y, and we obtain:

$$\mathcal{W} = \left[f(\alpha)xU^2yg(\beta)\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right].$$

Substituting $x \mapsto xU^*$, $y \mapsto U^*y$, one has:

$$\mathcal{W} = \left[f(\alpha)xyg(\beta)\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right].$$

Commuting back $f(\alpha)$ with x and $g(\beta)$ with y, we obtain:

$$\mathcal{W} = \left[xf(\alpha)g(\beta)y\,,\ f,g \in C_0(\mathbb{C}),\, x,y \in \lambda(C_0(\mathbb{C}^2))\right] = B.$$

This concludes the proof. ■

We will now find the commutation relations between $\hat{\alpha}$ and $\hat{\beta}$.

Proposition 2 *One has:*

1. *α et $T_l^* + T_r^*$ strongly commute and $\hat{\alpha} = e^{x(T_l^*+T_r^*)}\alpha$.*
2. *β et $T_l^* - T_r^*$ strongly commute and $\hat{\beta} = e^{x(T_l^*-T_r^*)}\beta$.*

Thus, the polar decompositions are given by :

$$\begin{aligned} Ph(\hat{\alpha}) &= e^{-ixIm(T_l+T_r)}Ph(\alpha)\,, \quad |\hat{\alpha}| = e^{xRe(T_l+T_r)}|\alpha|, \\ Ph(\hat{\beta}) &= e^{-ixIm(T_l-T_r)}Ph(\beta)\,, \quad |\hat{\beta}| = e^{xRe(T_l-T_r)}|\beta|. \end{aligned}$$

Moreover, we have the following relations:

1. *$|\hat{\alpha}|$ and $|\hat{\beta}|$ strongly commute,*
2. *$Ph(\hat{\alpha})Ph(\hat{\beta}) = Ph(\hat{\beta})Ph(\hat{\alpha})$,*
3. *$Ph(\hat{\alpha})|\hat{\beta}|Ph(\hat{\alpha})^* = e^{4x}|\hat{\beta}|$,*
4. *$Ph(\hat{\beta})|\hat{\alpha}|Ph(\hat{\beta})^* = e^{4x}|\hat{\alpha}|$.*

Proof. Using (8), we find, for all $z \in \mathbb{C}$:

$$e^{i\mathrm{Im}(z(T_l^*+T_r^*))}\alpha e^{-i\mathrm{Im}(z(T_l^*+T_r^*))} = \lambda_{-\overline{z},-\overline{z}}\alpha\lambda^*_{-\overline{z},-\overline{z}} = e^{-\overline{z}+\overline{z}}\alpha = \alpha.$$

Thus, $T_l^* + T_r^*$ and α strongly commute. Moreover, because $e^{ix\mathrm{Im}T_lT_l^*} = 1$, one has:

$$\hat{\alpha} = e^{-ix\mathrm{Im}T_lT_r^*}\alpha e^{ix\mathrm{Im}T_lT_r^*} = e^{-ix\mathrm{Im}T_l(T_l+T_r)^*}\alpha e^{ix\mathrm{Im}T_l(T_l+T_r)^*}.$$

We can now prove the point 1 using the equality $e^{-ix\mathrm{Im}T_l\omega}\alpha e^{ix\mathrm{Im}T_l\omega} = e^{x\omega}\alpha$, the preceding equation and the fact that $T_l^*+T_r^*$ and α strongly commute. The proof of the second assertion is similar and the polar decompositions follows. From (8) we deduce :

$$\begin{array}{rcl} e^{-ix\mathrm{Im}(T_r-T_l)}\alpha e^{ix\mathrm{Im}(T_r-T_l)} & = & e^{-2x}\alpha, \\ e^{ix\mathrm{Im}(T_l+T_r)}\beta e^{-ix\mathrm{Im}(T_l+T_r)} & = & e^{-2x}\beta, \\ e^{ix\mathrm{Re}(T_r-T_l)}\alpha e^{-ix\mathrm{Re}(T_r-T_l)} & = & e^{2ix}\alpha, \\ e^{ix\mathrm{Re}(T_l+T_r)}\beta e^{-ix\mathrm{Re}(T_l+T_r)} & = & e^{-2ix}\beta. \end{array}$$

It is now easy to prove the last relations from the preceding equations and the polar decompositions. ■

We can now give a formula for the comultiplication.

Proposition 3 *Let $\hat{\Delta}_x$ be the comultiplication on $\hat{A}_x$. One has:*

$$\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha},\ \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} \dot{+} \hat{\beta} \otimes \hat{\alpha}^{-1}.$$

Proof. Using the Preliminaries, we have that $\hat{\Delta}_x = \Upsilon\Gamma(.)\Upsilon^*$, where

$$\Upsilon = e^{ix\mathrm{Im}T_r \otimes T_l^*}$$

and Γ is given by

- $\Gamma(T_l) = T_l \otimes 1$, $\Gamma(T_r) = 1 \otimes T_r$;
- Γ restricted to $C_0(G)$ is equal to the comultiplication Δ_G.

Define $R = \Upsilon\Gamma(U^*)$. One has $\Delta_x(\hat{\alpha}) = R(\alpha \otimes \alpha)R^*$. Thus, it is sufficient to show that $(U \otimes U)R$ commute with $\alpha \otimes \alpha$. Indeed, in this case, one has

$$\hat{\Delta}_x(\hat{\alpha}) = R(\alpha\otimes\alpha)R^* = (U^*\otimes U^*)(U\otimes U)R(\alpha\otimes\alpha)R^*(U^*\otimes U^*)(U\otimes U) = \hat{\alpha}\otimes\hat{\alpha}.$$

Let us show that $(U \otimes U)R$ commute with $\alpha \otimes \alpha$. From the equality $U = e^{ix\mathrm{Im}T_l T_r^*}$, we deduce that

$$\Gamma(U^*) = e^{-ix\mathrm{Im}T_l \otimes T_r^*}, \quad U \otimes U = e^{ix\mathrm{Im}(T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^*)}.$$

Thus, $R = e^{-ix\mathrm{Im}(T_r^* \otimes T_l + T_l \otimes T_r^*)}$ and

$$(U \otimes U)R = e^{ix\mathrm{Im}(T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^* - T_r^* \otimes T_l - T_l \otimes T_r^*)}.$$

Notice that

$$T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^* - T_r^* \otimes T_l - T_l \otimes T_r^* = (T_l \otimes 1 - 1 \otimes T_l)(T_r^* \otimes 1 - 1 \otimes T_r^*).$$

Thus, it suffices to show that $T_l \otimes 1 - 1 \otimes T_l$ and $T_r^* \otimes 1 - 1 \otimes T_r^*$ strongly commute with $\alpha \otimes \alpha$. This follows from the equations

$$\begin{aligned} & e^{i\mathrm{Im}z(T_r^* \otimes 1 - 1 \otimes T_r^*)}(\alpha \otimes \alpha)e^{-i\mathrm{Im}z(T_r^* \otimes 1 - 1 \otimes T_r^*)} \\ & = (\lambda_{0,-\overline{z}} \otimes \lambda_{0,\overline{z}})(\alpha \otimes \alpha)(\lambda_{0,-\overline{z}} \otimes \lambda_{0,\overline{z}})^* \\ & = e^{-\overline{z}}e^{\overline{z}}\alpha \otimes \alpha = \alpha \otimes \alpha, \quad \forall z \in \mathbb{C} \end{aligned}$$

and

$$
\begin{aligned}
& e^{i\mathrm{Im}z(T_l\otimes 1-1\otimes T_l)}(\alpha\otimes\alpha)e^{-i\mathrm{Im}z(T_l\otimes 1-1\otimes T_l)} \\
& = (\lambda_{z,0}\otimes\lambda_{-z,0})(\alpha\otimes\alpha)(\lambda_{z,0}\otimes\lambda_{-z,0})^* \\
& = e^{-z}e^{z}\alpha\otimes\alpha = \alpha\otimes\alpha, \quad \forall z\in\mathbb{C}.
\end{aligned}
$$

Put $S = \Upsilon\Gamma(U)$. One has:

$$
\hat{\Delta}_x(\hat{\beta}) = S(\alpha\otimes\beta + \beta\otimes\alpha^{-1})S^* = S(\alpha\otimes\beta)S^* \dot{+} S(\beta\otimes\alpha^{-1})S^*.
$$

As before, we see that it suffices to show that $(U\otimes U^*)S$ commutes with $\alpha\otimes\beta$ and that $(U^*\otimes U)S$ commutes with $\beta\otimes\alpha^{-1}$, and one can check this in the same way . ■

Let us summarize the preceding results in the following corollary (see [16, 5] for the definition of commutation relation between unbounded operators):

Corollary 1 *Let $q = e^{8x}$. The C^*-algebra $\hat{A}_x$ is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ affiliated with $\hat{A}_x$ such that*

$$
\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \quad \hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha}.
$$

Moreover, the comultiplication $\hat{\Delta}_x$ is given by

$$
\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha}\otimes\hat{\alpha},\ \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha}\otimes\hat{\beta}\dot{+}\hat{\beta}\otimes\hat{\alpha}^{-1}.
$$

Remark. One can show, using the results of [4], that the application ($q\mapsto W_q$) which maps the parameter q to the multiplicative unitary of the twisted l.c. quantum group is continuous in the σ-weak topology.

4 Appendix

Let us cite some results on operators affiliated with a C^*-algebra.

Proposition 4 *Let $A\subset\mathcal{B}(H)$ be a non degenerated C^*-subalgebra and T a normal densely defined closed operator on H. Let $\mathcal{I}$ be the vector space generated by $f(T)a$, where $f\in C_0(\mathbb{C})$ and $a\in A$. Then:*

$$
(T\eta A) \Leftrightarrow \left(\begin{array}{c} f(T)\in M(A) \textit{ for any } f\in C_0(\mathbb{C}) \\ \textit{et } \mathcal{I} \textit{ is dense in } A \end{array} \right).
$$

Proof. If T is affiliated with A, then it is clear that $f(T)\in\mathrm{M}(A)$ for any $f\in C_0(\mathbb{C})$, and that $\mathcal{I}$ is dense in A (because $\mathcal{I}$ contains $(1+T^*T)^{-\frac{1}{2}}A$). To show the converse, consider the *-homomorphism $\pi_T : C_0(\mathbb{C})\to\mathrm{M}(A)$ given by $\pi_T(f) = f(T)$. By hypothesis, $\pi_T(C_0(\mathbb{C}))A$ is dense in A. So, $\pi_T\in\mathrm{Mor}(C_0(\mathbb{C}), A)$ and $T = \pi_T(z\mapsto z)$ is then affiliated with A. ■

Proposition 5 *Let $A \subset \mathcal{B}(H)$ be a non degenerated C^*-subalgebra and $T_1, T_2, \ldots, T_N$ normal operators affiliated with A. Let us denote by $\mathcal{V}$ the vector space generated by the products of the form $f_1(T_1)f_2(T_2)\ldots f_N(T_N)$, with $f_i \in C_0(\mathbb{C})$. If $\mathcal{V}$ is a dense vector subspace of A, then A is generated by $T_1, T_2, \ldots, T_N$.*

Proof. This follows from Theorem 3.3 in [15]. ∎

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